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A semilocal convergence result for Newton's method under generalized conditions of Kantorovich

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ABSTRACT

From Kantorovich's theory we establish a general semilocal convergence result for Newton's method based fundamentally on a generalization required to the second derivative of the operator involved. As a consequence, we obtain a modification of the domain of starting points for Newton's method and improve the a priori error estimates. Finally, we illustrate our study with an application to a special case of conservative problems.

1. Introduction

It is well-known that solving equations of the form $F(x) = 0$ where F is a nonlinear operator, $F : \Omega \subseteq X \to Y$, defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y, is a very common problem in engineering and science. Although some equations can be solved analytically, we usually look for numerical approximations of the solutions, since finding exact solutions is usually difficult. To approximate a solution of $F(x) = 0$ we normally use iterative methods and Newton's method,

$$\begin{cases} x_0 \in \Omega, \\ x_n = x_{n-1} - [F'(x_{n-1})]^{-1} F(x_{n-1}), \quad n \in \mathbb{N}, \end{cases} \tag{1}$$

is one of the most used because of its simplicity, easy implementation and efficiency.

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The first semilocal convergence result for Newton's method in Banach spaces is due to L.V. Kantorovich, which is usually known as the Newton–Kantorovich theorem [12] and is proved under the following conditions for the operator F and the starting point x_0:

(C_1) There exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ for some $x_0 \in \Omega$, $\|\Gamma_0\| \leq \beta$ and $\|\Gamma_0 F(x_0)\| \leq \eta$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y to X,

(C_2) $\|F''(x)\| \leq \ell$, for $x \in \Omega$,

(C_3) $\ell\beta\eta \leq \frac{1}{2}$ and $B\left(x_0, \frac{1-\sqrt{1-2\ell\beta\eta}}{\ell\beta}\right) \subset \Omega$.

Since then, many papers have appeared that study the semilocal convergence of the method. Most of them are modifications of the Newton–Kantorovich theorem in order to relax conditions (C_1)–(C_3), specially condition (C_2). In [6], we present a first generalization of condition (C_2), where we replace (C_2) by the condition

$$\|F''(x)\| \leq \omega(\|x\|), \quad x \in \Omega, \tag{2}$$

where $\omega : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ is a non-decreasing continuous real function. Obviously, condition (2) generalizes condition (C_2). From the function ω, we construct a scalar function that allows us to define a majorizing sequence of Newton's method in Banach spaces, so that the convergence of Newton's method in Banach spaces is guaranteed from the majorizing sequence. The main aim of this paper is to generalize condition (2) to successive derivatives of the operator F.

The paper begins in Section 2 by recalling the concept of majorizing sequence and presenting the Newton–Kantorovich theorem. In this section we also introduce the new convergence conditions for the general case. In Section 3, we present a new general semilocal convergence theorem for Newton's method and indicate how the majorizing sequences are constructed. We also include information about the existence and uniqueness of solution and a result on the a priori error estimates that leads to the quadratic convergence of Newton's method. In Section 4, we give a particular case of our general result. Finally, in Section 5, we present an application where a conservative problem is involved. We clearly show the advantages of our new semilocal convergence result with respect to the Newton–Kantorovich theorem.

Throughout the paper we denote $\overline{B(x, \varrho)} = \{y \in X; \|y - x\| \leq \varrho\}$ and $B(x, \varrho) = \{y \in X; \|y - x\| < \varrho\}$.

2. Preliminary information

The known Newton–Kantorovich theorem [12] guarantees the semilocal convergence of Newton's method in Banach spaces and gives a priori error estimates and information about the existence and uniqueness of the solution. Kantorovich proves the theorem by using two different techniques [10,11], although the most prominent one is the majorant principle [11], which is based on the concept of majorizing sequence. This technique has been usually used later by other authors to analyze the semilocal convergence of several iterative methods [2,1,4,16]. We begin by introducing the concept of majorizing sequence and remembering how it is used to prove the convergence of sequences in Banach spaces.

Definition 1. If $\{x_n\}$ is a sequence in a Banach space X and $\{t_n\}$ is a scalar sequence, then $\{t_n\}$ is a *majorizing sequence* of $\{x_n\}$ if $\|x_n - x_{n-1}\| \leq t_n - t_{n-1}$, for all $n \in \mathbb{N}$.

Observe, from the last inequality, it follows the sequence $\{t_n\}$ is non-decreasing. The interest of the majorizing sequence is that the convergence of the sequence $\{x_n\}$ in the Banach space X is deduced from the convergence of the scalar sequence $\{t_n\}$, as we can see in the following result [12]:

Lemma 2. *Let $\{x_n\}$ be a sequence in a Banach space X and $\{t_n\}$ a majorizing sequence of $\{x_n\}$. Then, if $\{t_n\}$ converges to $t^* < \infty$, there exists $x^* \in X$ such that $x^* = \lim_n x_n$ and $\|x^* - x_n\| \leq t^* - t_n$, for $n = 0, 1, 2, \ldots$.*

Once the concept of majorizing sequence is introduced, we can already establish Kantorovich's theory for Newton's method. In [12], Kantorovich proves the semilocal convergence of Newton's method under conditions (C_1)–(C_3). For this, Kantorovich first considers that $F \in \mathcal{C}^{(2)}(\Omega_0)$, with $\Omega_0 = \overline{B(x_0, r_0)} \subseteq X$, requires the existence of a real auxiliary function $f \in \mathcal{C}^{(2)}([s_0, s'])$, with $s' - s_0 \le r_0$, and proves the following general semilocal convergence for Newton's method.

Theorem 3 (*The General Semilocal Convergence Theorem*)**.** *Let $F : \Omega_0 \subseteq X \to Y$ be a twice continuously differentiable operator defined on a non-empty open convex domain $\Omega_0 = \overline{B(x_0, r_0)}$ of a Banach space X with values in a Banach space Y. Let $f \in \mathcal{C}^{(2)}([s_0, s'])$ be a scalar function. Suppose that the following conditions are satisfied:*

(K_1) *There exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ for some $x_0 \in \Omega$, $\|\Gamma_0\| \le -\frac{1}{f'(s_0)}$ and $\|\Gamma_0 F(x_0)\| \le -\frac{f(s_0)}{f'(s_0)}$,*
(K_2) *$\|F''(x)\| \le f''(s)$ if $\|x - x_0\| \le s - s_0 \le r_0$,*
(K_3) *the equation $f(s) = 0$ has a solution in $[s_0, s']$.*

Then, Newton's sequence, given by (1), converges to a solution x^* of the equation $F(x) = 0$, starting at x_0. Moreover,

$$\|x^* - x_n\| \le s^* - s_n, \quad n \ge 0,$$

where s^* is the smallest solution of the equation

$$f(s) = 0 \tag{3}$$

in $[s_0, s']$. Furthermore, if $f(s') \le 0$ and (3) has a unique solution in $[s_0, s']$, then x^* is the unique solution of $F(x) = 0$ in Ω_0.

According to the convergence conditions required, the last result is known as the general semilocal convergence theorem of Newton's method for operators with second derivative bounded in norm.

In practice, the application of the last theorem is complicated, since the scalar function f is unknown. So, the following result is of particular interest, since if the operator F satisfies (C_2), from (K_1) and (K_2) and solving the corresponding problem of interpolation fitting, we can consider the polynomial

$$p(s) = \frac{\ell}{2}(s - s_0)^2 - \frac{s - s_0}{\beta} + \frac{\eta}{\beta} \tag{4}$$

as the scalar function f in Theorem 3. In this case, we obtain the classic Newton–Kantorovich theorem, that establishes the semilocal convergence of Newton's method under conditions (C_1)–(C_3), that we call classic conditions of Kantorovich (not to be confused with the general conditions of Kantorovich, which are conditions (K_1)–(K_3) of Theorem 3).

Theorem 4 (*The Newton–Kantorovich Theorem*)**.** *Let $F : \Omega \subseteq X \to Y$ be a twice continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. Suppose that conditions (C_1)–(C_3) hold and $\overline{B(x_0, s^* - s_0)} \subseteq \Omega$, where $s^* = s_0 + \frac{1-\sqrt{1-2\ell\beta\eta}}{\ell\beta}$. Then Newton's sequence, given by (1), converges to a solution x^* of the equation $F(x) = 0$, starting at x_0, and $x_n, x^* \in \overline{B(x_0, s^* - s_0)}$, for all $n = 0, 1, 2, \ldots$. Moreover, if $\ell\beta\eta < \frac{1}{2}$, x^* is the unique solution of $F(x) = 0$ in $B(x_0, s^{**} - s_0) \cap \Omega$, where $s^{**} = s_0 + \frac{1+\sqrt{1-2\ell\beta\eta}}{\ell\beta}$, and if $\ell\beta\eta = \frac{1}{2}$, x^* is unique in $\overline{B(x_0, s^* - s_0)}$. Furthermore,*

$$\|x_{n+1} - x_n\| \le s_{n+1} - s_n \quad \textit{and} \quad \|x^* - x_n\| \le s^* - s_n, \quad \textit{for all } 0, 1, 2, \ldots,$$

where $s_n = s_{n-1} - \frac{p(s_{n-1})}{p'(s_{n-1})}$, with $n \in \mathbb{N}$ and $p(s)$ is defined in (4).

Note that polynomial (4) can be obtained otherwise, without interpolation fitting, by solving the following initial value problem:

$$\begin{cases} p''(s) = \ell, \\ p(s_0) = \dfrac{\eta}{\beta}, \qquad p'(s_0) = -\dfrac{1}{\beta}. \end{cases}$$

Observe that Kantorovich's polynomial (4) is such that

$$p(s+s_0) = \hat{p}(s), \quad \text{where } \hat{p}(s) = \frac{\ell}{2}s^2 - \frac{s}{\beta} + \frac{\eta}{\beta}.$$

Therefore, the scalar sequences given by Newton's method with $p(s)$ and $\hat{p}(s)$ can be obtained, one from the other, by translation. As a consequence, the last result is independent of the value s_0. For this reason, we always choose $s_0 = 0$, which simplifies considerably the expressions used.

This new way of getting polynomial (4) has the advantage of being able to be generalized to other conditions, so that we can then construct the scalar function f under more general conditions, as we will see later.

3. Main results

As we have indicated in the introduction, condition (C_2) limits the application of the classic Newton–Kantorovich theorem. In Section 5, we show the last with an example. The main idea of this paper is first to generalize Kantorovich's conditions by modifying (K_1) and (K_2). For this, as in general Theorem 3, we construct a scalar function $f \in \mathcal{C}^{(k)}([t_0, t'])$, with $t_0, t' \in \mathbb{R}$, $k \geq 3$, that satisfies:

$(\widetilde{K_1})$ There exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ for some $x_0 \in \Omega$, $\|\Gamma_0\| \leq -\frac{1}{f'(t_0)}$, $\|\Gamma_0 F(x_0)\| \leq -\frac{f(t_0)}{f'(t_0)}$ and $\|F^{(i)}(x_0)\| \leq f^{(i)}(t_0)$, with $i = 2, 3, \ldots, k-1$ and $k \geq 3$,

$(\widetilde{K_2})$ $\|F^{(k)}(x)\| \leq f^{(k)}(t)$, for $\|x - x_0\| \leq t - t_0$, $x \in \Omega$ and $t \in [t_0, t']$.

Next, we prove the semilocal convergence of Newton's method under conditions $(\widetilde{K_1})$ and $(\widetilde{K_2})$, by using the majorant principle, as Kantorovich does in [12] under conditions (K_1) and (K_2). For this, we construct a majorizing sequence $\{t_n\}$ of Newton's sequence $\{x_n\}$ in the Banach space X. To obtain the sequence $\{t_n\}$ we use a real function $f(t)$ defined in $[t_0, t'] \subset \mathbb{R}$ as follows:

$$t_0 \text{ given}, \quad t_n = N(t_{n-1}) = t_{n-1} - \frac{f(t_{n-1})}{f'(t_{n-1})}, \quad n \in \mathbb{N}. \tag{5}$$

3.1. Semilocal convergence

As we can see in the classic Newton–Kantorovich theorem, the majorizing sequence $\{s_n\}$ that is constructed from Newton's method and using polynomial (4) with $s_0 = 0$, converges to the smallest positive solution s^* of the equation $p(s) = 0$. This convergence is clear, since polynomial (4) is a decreasing and convex function in $[s_0, s']$. By analogy with Kantorovich, if we want to apply the majorant principle to our particular situation, then the equation $f(t) = 0$ must have at least one solution t^* greater than t_0, so that the corresponding majorizing sequence $\{t_n\}$, under conditions $(\widetilde{K_1})$ and $(\widetilde{K_2})$, converges to t^* from t_0. Clearly, the first thing we need is to study the function f. Then, we give some properties of this function.

Theorem 5. *Let $f \in \mathcal{C}^{(k)}([t_0, t'])$ be with $t_0, t' \in \mathbb{R}$, $k \geq 3$, such that $f(t_0) > 0$, $f'(t_0) < 0$, $f^{(i)}(t_0) > 0$, for $i = 2, 3, \ldots, k-1$, and $f^{(k)}(t) > 0$.*

(a) *If there exists a solution α of $f'(t) = 0$ such that $\alpha > t_0$, then α is the unique minimum of $f(t)$ in $[t_0, t']$ and $f(t)$ is decreasing in $[t_0, \alpha)$.*

(b) *If $f(\alpha) \leq 0$, then $f(t) = 0$ has at least one solution in $[t_0, +\infty)$. Moreover, if t^* is the smallest solution of $f(t) = 0$ in $[t_0, +\infty)$, then $t_0 < t^* \leq \alpha$.*

As we are interested in the fact that (5) is a majorizing sequence of the sequence $\{x_n\}$ defined in (1), we establish the convergence of $\{t_n\}$ in the next result.

Theorem 6. *Let $\{t_n\}$ be the scalar sequence given in (5) with $f \in \mathcal{C}^{(k)}([t_0, t'])$ and $t_0, t' \in \mathbb{R}$, $k \geq 3$. Suppose that conditions $(\widetilde{K_1})$ and $(\widetilde{K_2})$ hold and there exists a solution $\alpha \in (t_0, t')$ of $f'(t) = 0$ such that $f(\alpha) \leq 0$. Then, (5) is a non-decreasing sequence that converges to t^*.*

Proof. As $f(t_0) > 0$, then $t_0 - t^* \leq 0$. By the mean value theorem, we obtain

$$t_1 - t^* = N(t_0) - N(t^*) = N'(\theta_0)(t_0 - t^*) \quad \text{with } \theta_0 \in (t_0, t^*),$$

so that $t_1 < t^*$, since $N'(t) = \frac{f(t)f''(t)}{f'(t)^2} > 0$ in $[t_0, t^*)$.

On the other hand, we have

$$t_1 - t_0 = -\frac{f(t_0)}{f'(t_0)} \geq 0.$$

By mathematical induction on n, we obtain $t_n < t^*$ and $t_n - t_{n-1} \geq 0$, since $(t_{n-1}, t^*) \subset (t_0, t^*)$. Therefore, we infer that sequence (5) converges to $r \in [t_0, t^*]$. Moreover, since t^* is the unique root of $f(t) = 0$ in $[t_0, t^*]$, it follows that $r = t^*$. ■

Next, from the following lemma, we see that (5) is a majorizing sequence of sequence (1) in the Banach space X.

Lemma 7. *Suppose that conditions $(\widetilde{K_1})$ and $(\widetilde{K_2})$ are satisfied and $f \in \mathcal{C}^{(k)}([t_0, t'])$ with $t_0, t' \in \mathbb{R}$. Suppose also that $x_n \in \Omega$, for all $n \geq 0$, and $f(\alpha) \leq 0$, where α is a solution of $f'(t) = 0$ in (t_0, t'). Then, for all $n \geq 1$, we have*

(i_n) $\|\Gamma_n\| \leq -\frac{1}{f'(t_n)}$,

(ii_n) $\|F^{(i)}(x_n)\| \leq f^{(i)}(t_n)$, *for* $i = 2, 3, \ldots, k-1$,

(iii_n) $\|F(x_n)\| \leq f(t_n)$,

(iv_n) $\|x_{n+1} - x_n\| \leq t_{n+1} - t_n$.

Proof. We prove (i_n)–(iv_n) by mathematical induction on n. As the cases $n = 1$ and the inductive step are proved in the same way, we only write the inductive step. We then suppose that (i_n)–(iv_n) are true for $n = 1, 2, \ldots, p-1$ and prove that they are also true for $n = p$.

(i_p): From Taylor's series we have

$$\begin{aligned} I - \Gamma_{p-1}F'(x_p) = I - \Gamma_{p-1}\Bigg(&\sum_{i=1}^{k-1} \frac{1}{(i-1)!} F^{(i)}(x_{p-1})(x_p - x_{p-1})^{i-1} \\ &+ \frac{1}{(k-2)!}\int_{x_{p-1}}^{x_p} F^{(k)}(x)(x_p - x)^{k-2}\,dx\Bigg). \end{aligned}$$

If we now denote $x = x_{p-1} + \tau(x_p - x_{p-1})$ and $t = t_{p-1} + \tau(t_p - t_{p-1})$ with $\tau \in [0, 1]$, then

$$\begin{aligned} \|x - x_0\| &\leq \tau\|x_p - x_{p-1}\| + \|x_{p-1} - x_{p-2}\| + \cdots + \|x_1 - x_0\| \\ &\leq \tau(t_p - t_{p-1}) + t_{p-1} - t_{p-2} + \cdots + t_1 - t_0 \\ &= t - t_0. \end{aligned}$$

Consequently,

$$\begin{aligned} \|I - \Gamma_{p-1}F'(x_p)\| \leq \|\Gamma_{p-1}\|\Bigg(&\sum_{i=2}^{k-1} \frac{1}{(i-1)!}\|F^{(i)}(x_{p-1})\|\|x_p - x_{p-1}\|^{i-1} \\ &+ \frac{1}{(k-2)!}\int_0^1 \left\|F^{(k)}\Big(x_{p-1} + \tau(x_p - x_{p-1})\Big)\right\| \\ &\times (1-\tau)^{k-2}\|x_p - x_{p-1}\|^{k-1}\,d\tau\Bigg) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{f'(t_{p-1})}\left(\sum_{i=2}^{k-1}\frac{1}{(i-1)!}f^{(i)}(t_{p-1})(t_p-t_{p-1})^{i-1}\right.\\
&\quad \left.+\frac{1}{(k-2)!}\int_{t_{p-1}}^{t_p} f^{(k)}(t)(t_p-t)^{k-2}\,dt\right)\\
&= 1-\frac{f'(t_p)}{f'(t_{p-1})} < 1.
\end{aligned}
$$

Therefore, by the Banach lemma on invertible operators, Γ_p exists and $\|\Gamma_p\| \le -\frac{1}{f'(t_p)}$.

(ii_p): For any $i \in \{2, 3, \dots, k-1\}$ and Taylor's series, we have

$$
\begin{aligned}
F^{(i)}(x_p) &= F^{(i)}(x_{p-1}) + F^{(i+1)}(x_{p-1})(x_p-x_{p-1}) + \frac{1}{2!}F^{(i+2)}(x_{p-1})(x_p-x_{p-1})^2\\
&\quad + \cdots + \frac{1}{(k-1-i)!}F^{(k-1)}(x_{p-1})(x_p-x_{p-1})^{k-1-i}\\
&\quad + \frac{1}{(k-1-i)!}\int_{x_{p-1}}^{x^p} F^{(k)}(x)(x_p-x)^{k-1-i}\,dx.
\end{aligned}
$$

Moreover, as $\|x-x_0\| \le t-t_0$ and $\|x_p-x_{p-1}\| \le t_p-t_{p-1}$, it follows that

$$
\begin{aligned}
\|F^{(i)}(x_p)\| &= \sum_{j=1}^{k-i}\frac{1}{(j-1)!}\|F^{(j+i-1)}(x_{p-1})\|\|x_p-x_{p-1}\|^{j-1}\\
&\quad + \frac{1}{(k-1-i)!}\int_0^1 \left\|F^{(k)}\Big(x_{p-1}+\tau(x_p-x_{p-1})\Big)\right\|(1-\tau)^{k-2}\|x_p-x_{p-1}\|^{k-i}\,d\tau\\
&= \sum_{j=1}^{k-i}\frac{1}{(j-1)!}f^{(j+i-1)}(t_{p-1})(t_p-t_{p-1})^{j-1} + \frac{1}{(k-1-i)!}\int_{t_{p-1}}^{t_p} f^{(k)}(t)(t_p-t)^{k-i-1}\,dt\\
&= f^{(i)}(t_p).
\end{aligned}
$$

(iii_p): From Taylor's series we have

$$
\begin{aligned}
F(x_p) &= \sum_{j=2}^{k-1}\frac{1}{j!}F^{(j)}(x_{p-1})(x_p-x_{p-1})^j\\
&\quad + \frac{1}{(k-1)!}\int_0^1 F^{(k)}\Big(x_{p-1}+t(x_p-x_{p-1})\Big)(1-t)^{k-1}(x_p-x_{p-1})^k\,dt,
\end{aligned}
$$

since $F(x_{p-1}) + F'(x_{p-1})(x_p-x_{p-1}) = 0$. As a consequence,

$$
\begin{aligned}
\|F(x_p)\| &\le \sum_{j=2}^{k-1}\frac{1}{j!}f^{(j)}(t_{p-1})(t_p-t_{p-1})^j\\
&\quad + \frac{1}{(k-1)!}\int_0^1 f^{(k)}\Big(t_{p-1}+t(t_p-t_{p-1})\Big)(1-t)^{k-1}(t_p-t_{p-1})^k\,dt\\
&= f(t_p).
\end{aligned}
$$

(iv_p): $\|x_{p+1}-x_p\| \le \|\Gamma_p\|\|F(x_p)\| \le -\frac{f(t_p)}{f'(t_p)} = t_{p+1}-t_p$.

The proof is complete. ■

Remark 8. From $(\widetilde{K_1})$ and $(\widetilde{K_2})$, items (i_n), (ii_n) and (iv_n) are obvious if $n = 0$. Item (iii_n) with $n = 0$ does not need to prove (iv_n) with $n = 0$, since it follows from the condition $\|\Gamma_0 F(x_0)\| \le \eta$.

Once we have seen that (5) is a majorizing sequence of sequence (1), we are ready to prove the semilocal convergence of (1) in the Banach space X.

Theorem 9. *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a nonlinear k ($k \geq 3$) times continuously differentiable operator on a non-empty open convex domain Ω and $f \in \mathcal{C}^{(k)}([t_0, t'])$ with $t_0, t' \in \mathbb{R}$. Suppose that $(\widetilde{K_1})$ and $(\widetilde{K_2})$ hold, there exists $\alpha \in (t_0, t')$ such that $f'(\alpha) = 0$ and $f(\alpha) \leq 0$, and $\overline{B(x_0, t^* - t_0)} \subseteq \Omega$. Then, Newton's sequence $\{x_n\}$, given by (1), converges to a solution x^* of $F(x) = 0$ starting at x_0. Moreover, $x_n, x^* \in \overline{B(x_0, t^* - t_0)}$ and*

$$\|x^* - x_n\| \leq t^* - t_n, \quad \text{for all } n = 0, 1, 2, \ldots,$$

where $\{t_n\}$ is defined in (5).

Proof. From $(\widetilde{K_1})$ and $(\widetilde{K_2})$, it is clear that x_1 is well-defined and $\|x_1 - x_0\| = t_1 - t_0 < t^* - t_0$, so that $x_1 \in B(x_0, t^* - t_0) \subset \Omega$.

We now suppose that x_n is well-defined and $x_n \in B(x_0, t^* - t_0)$ for $n = 1, 2, \ldots, p - 1$.

After that, by the last lemma, the operator Γ_{p-1} exists and $\|\Gamma_{p-1}\| \leq -\frac{1}{f'(t_{p-1})}$. In addition, x_p is well-defined. Moreover, since $\|x_{n+1} - x_n\| \leq t_{n+1} - t_n$ for $n = 1, 2, \ldots, p - 1$, we have

$$\|x_p - x_0\| \leq \sum_{i=0}^{p-1} \|x_{i+1} - x_i\| \leq \sum_{i=0}^{p-1} (t_{i+1} - t_i) = t_p - t_0 < t^* - t_0,$$

so that $x_p \in B(x_0, t^* - t_0)$. Therefore, Newton's sequence $\{x_n\}$ is well-defined and $x_n \in B(x_0, t^* - t_0)$ for all $n \geq 0$.

By the last lemma, we also have that $\|F(x_n)\| \leq f(t_n)$ and

$$\|x_{n+1} - x_n\| \leq \|\Gamma_n\| \|F(x_n)\| \leq -\frac{f(t_n)}{f'(t_n)} = t_{n+1} - t_n,$$

for all $n \geq 0$, so that $\{t_n\}$ is a majorizing sequence of $\{x_n\}$ and is convergent. Moreover, as $\lim_{n\to+\infty} t_n = t^*$, if $x^* = \lim_{n\to+\infty} x_n$, then $\|x^* - x_n\| \leq t^* - t_n$, for all $n = 0, 1, 2, \ldots$. Furthermore, as $\|F(x_n)\| \leq f(t_n)$, for all $n = 0, 1, 2, \ldots$, then by letting $n \to +\infty$, it follows $F(x^*) = 0$ by the continuity of F. ■

3.2. Uniqueness of solution

After proving the semilocal convergence of Newton's method and locating the solution x^*, we prove the uniqueness of x^*. Before, we give the following technical lemma that is used later.

Lemma 10. *If the scalar function $f(t)$ has two real zeros t^* and t^{**} such that $t_0 < t^* \leq t^{**}$, the conditions of the last theorem are satisfied and $x \in \overline{B(x_0, t^{**} - t_0)} \cap \Omega$, then*

$$\|F''(x)\| \leq f''(t), \quad \text{for } \|x - x_0\| \leq t - t_0.$$

Proof. From Taylor's series, it follows

$$F''(x) = \sum_{i=2}^{k-1} \frac{1}{(i-2)!} F^{(i)}(x_0)(x - x_0)^{i-2} + \frac{1}{(k-3)!} \int_0^1 F^{(k)}\Big(x_0 + \tau(x - x_0)\Big)(1 - \tau)^{k-3}(x - x_0)^{k-2}\, d\tau.$$

If we take norms and take into account (K_1) and (K_2), for $\|x - x_0\| \leq t - t_0$, we have

$$\|F''(x)\| \leq \sum_{i=2}^{k-1} \frac{1}{(i-2)!} \|F^{(i)}(x_0)\| \|x - x_0\|^{i-2}$$

$$
\begin{aligned}
&+\frac{1}{(k-3)!}\int_0^1 \left\|F^{(k)}\Big(x_0+\tau(x-x_0)\Big)\right\|(1-\tau)^{k-3}\|x-x_0\|^{k-2}\,d\tau\\
&\le \sum_{i=2}^{k-1}\frac{1}{(i-2)!}f^{(i)}(t_0)(t-t_0)^{i-2}+\frac{1}{(k-3)!}\int_{t_0}^{t} f^{(k)}(s)(t-s)^{k-3}\,ds\\
&= f''(t),
\end{aligned}
$$

since $z = x_0 + \tau(x - x_0)$ and $s = t_0 + \tau(t - t_0)$ with $\tau \in [0, 1]$ and $\|z - x_0\| \le \tau\|x - x_0\| \le \tau - (t - t_0) = s - t_0$. ■

Note that if $f(t)$ has two real zeros t^* and t^{**} such that $t_0 < t^* \le t^{**}$, then the uniqueness of solution follows from the next theorem.

Theorem 11. *Under the conditions of the last theorem, if the scalar function $f(t)$ has two real zeros t^* and t^{**} such that $t_0 < t^* \le t^{**}$, then the solution x^* is unique in $B(x_0, t^{**} - t_0) \cap \Omega$ if $t^* < t^{**}$ or in $\overline{B(x_0, t^* - t_0)}$ if $t^{**} = t^*$.*

Proof. Suppose that $t^* < t^{**}$ and y^* is another solution of $F(x) = 0$ in $B(x_0, t^{**} - t_0) \cap \Omega$. Then,

$$\|y^* - x_0\| \le \rho(t^{**} - t_0) \quad \text{with } \rho \in (0, 1).$$

We now suppose that $\|y^* - x_k\| \le \rho^{2^k}(t^{**} - t_k)$ for $k = 0, 1, \dots, n$. In addition,

$$
\begin{aligned}
\|y^* - x_{n+1}\| &= \left\|-\Gamma_n\left(F(y^*) - F(x_n) - F'(x_n)(y^* - x_n)\right)\right\|\\
&= \left\|-\Gamma_n\int_0^1 F''\left(x_n+\tau(y^*-x_n)\right)(1-\tau)(y^*-x_n)^2\,d\tau\right\|.
\end{aligned}
$$

As $\|x_n + \tau(y^* - x_n) - x_0\| \le t_n + \tau(t^{**} - t_n) - t_0$, it follows that

$$\|y^* - x_{n+1}\| \le -\frac{\mu}{f'(t_n)}\|y^* - x_n\|^2,$$

where $\mu = \int_0^1 \left(f''\left(t_n + \tau(t^{**} - t_n)\right)\right)(1 - \tau)\,d\tau$.

On the other hand, we also have

$$t^{**} - t_{n+1} = -\frac{1}{f'(t_n)}\int_{t_n}^{t^{**}} f''(z)(t^{**} - z)\,dz = -\frac{\mu}{f'(t_n)}(t^{**} - t_n)^2.$$

Therefore,

$$\|y^* - x_{n+1}\| \le \frac{t^{**} - t_{n+1}}{(t^{**} - t_n)^2}\|y^* - x_n\|^2 \le \rho^{2^{n+1}}(t^{**} - t_{n+1}),$$

so that $y^* = x^*$.

If $t^{**} = t^*$ and y^* is another solution of $F(x) = 0$ in $\overline{B(x_0, t^{**} - t_0)}$, then $\|y^* - x_0\| \le t^* - t_0$. Proceeding similarly to the previous case, we can prove by mathematical induction on n that $\|y^* - x_n\| \le t^{**} - t_n$. Since $t^{**} = t^*$ and $\lim_{n\to+\infty} t_n = t^*$, the uniqueness of the solution is now easy to follow. ■

Note that the last theorem is a generalization of that obtained under Kantorovich's conditions.

3.3. Quadratic convergence of Newton's method

In this section we see the quadratic convergence of Newton's method under conditions $(\widetilde{K_1})$ and $(\widetilde{K_2})$. Notice first that if $f(t)$ has two real zeros t^* and t^{**} such that $t_0 < t^* \le t^{**}$, we can then write

$$f(t) = (t^* - t)(t^{**} - t)g(t)$$

with $g(t^*) \ne 0$ and $g(t^{**}) \ne 0$. Next, from Ostrowski's technique [13], we give a theorem which provides some a priori error estimates for Newton's method. The proof of the theorem is analogous to that given in [6].

Theorem 12. *Let $f \in \mathcal{C}^{(k)}([t_0, t'])$ with $t_0, t' \in \mathbb{R}$, $k \geq 3$. Suppose that conditions $(\widetilde{K_1})$ and $(\widetilde{K_2})$ are satisfied and $f(t)$ has two real zeros t^* and t^{**} such that $t_0 < t^* \leq t^{**}$.*

(a) *If $t^* < t^{**}$, then*

$$\frac{(t^{**}-t^*)\theta^{2^n}}{\sqrt{m_1}-\theta^{2^n}} < t^* - t_n < \frac{(t^{**}-t^*)\Delta^{2^n}}{\sqrt{M_1}-\Delta^{2^n}}, \quad n \geq 0,$$

where $\theta = \frac{t^*}{t^{**}}\sqrt{m_1}$, $\Delta = \frac{t^*}{t^{**}}\sqrt{M_1}$, $m_1 = \min\{H_1(t); t \in [0, t^*]\}$, $M_1 = \max\{H_1(t); t \in [0, t^*]\}$, $H_1(t) = \frac{(t^{**}-t)g'(t)-g(t)}{(t^*-t)g'(t)-g(t)}$ and provided that $\theta < 1$ and $\Delta < 1$.

(b) *If $t^* = t^{**}$, then*

$$m_2^n t^* \leq t^* - t_n \leq M_2^n t^*,$$

where $m_2 = \min\{H_2(t); t \in [0, t^*]\}$, $M_2 = \max\{H_2(t); t \in [0, t^*]\}$, $H_2(t) = \frac{(t^*-t)g'(t)-g(t)}{(t^*-t)g'(t)-2g(t)}$ and provided that $m_2 < 1$ and $M_2 < 1$.

From the last theorem, it follows that the convergence of Newton's method is quadratic if $t^* < t^{**}$ and linear if $t^* = t^{**}$.

4. A particular semilocal convergence result

On one hand, we remember that Kantorovich solves a problem of interpolation fitting to obtain the real function $f(t)$ from which the majorizing sequence $\{t_n\}$ is defined. In particular, Kantorovich considers that $f(t)$ is a second degree polynomial, fits its coefficients with conditions (C_1)–(C_2) and obtains polynomial (4).

On the other hand, if we consider that

$(\widetilde{C_1})$ There exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ for some $x_0 \in \Omega$, $\|\Gamma_0\| \leq \beta$, $\|\Gamma_0 F(x_0)\| \leq \eta$ and $\|F^{(i)}(x_0)\| \leq M_i$, with $i = 2, 3, \ldots, k-1$ and $k \geq 3$,

$(\widetilde{C_2})$ $\|F^{(k)}(x)\| \leq \omega(\|x\|)$ for $\|x - x_0\| \leq t - t_0$, where $\omega : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ is a non-decreasing continuous function,

we cannot obtain a real function by interpolation fitting, since $(\widetilde{C_2})$ does not allow determining the class of functions where $(\widetilde{C_1})$ can be applied.

Now, taking into account that $\|x\| \leq \|x_0\| + \|x - x_0\|$, from $(\widetilde{C_2})$, it follows

$$\|F^{(k)}(x)\| \leq \omega(\|x\|) \leq \omega(t - t_0 + \|x_0\|).$$

From now on, we denote $\omega(t; t_0) = \omega(t - t_0 + \|x_0\|)$. So, condition $(\widetilde{C_2})$ is written as

$$\|F^{(k)}(x)\| \leq \omega(t; t_0) \quad \text{for } \|x - x_0\| \leq t - t_0,$$

where $\omega : [t_0, +\infty) \to \mathbb{R}$ is continuous, non-decreasing and such that $\omega(t_0; t_0) \geq 0$.

In addition, to find a real function f from conditions $(\widetilde{C_1})$ and $(\widetilde{C_2})$, it is enough to solve the following initial value problem:

$$\begin{cases} y^{(k)}(t) = \omega(t; t_0), \\ y(t_0) = \dfrac{\eta}{\beta}, \quad y'(t_0) = -\dfrac{1}{\beta}, \\ y''(t_0) = M_2, \quad y'''(t_0) = M_3, \quad \ldots, \quad y^{(k-1)}(t_0) = M_{k-1}, \end{cases}$$

whose solution function is given in the next result.

Theorem 13. *For any real numbers $\beta \neq 0$, $\eta, M_2, M_3, \ldots, M_{k-1}$, there exists only one solution $f(t)$ of the last initial value problem in $[t_0, t']$, that is,*

$$\begin{aligned} f(t) = {} & \int_{t_0}^{t}\int_{t_0}^{\theta_{k-1}}\cdots\int_{t_0}^{\theta_1} \omega(z; t_0)\, dz\, d\theta_1 \ldots d\theta_{k-1} \\ & + \frac{M_{k-1}}{(k-1)!}(t-t_0)^{k-1} + \cdots + \frac{M_2}{2!}(t-t_0)^2 - \frac{t-t_0}{\beta} + \frac{\eta}{\beta}. \end{aligned} \tag{6}$$

Observe that (6) is reduced to a polynomial of degree k if ω is constant (see [5,7]).

To apply Theorem 9, the equation $f(t) = 0$ must have at least one solution greater than t_0, so that we have to guarantee the convergence of the scalar sequence $\{t_n\}$, from t_0, to this solution. We then study the function $f(t)$ defined in (6).

Theorem 14. *Let f and ω be the functions defined respectively in (6) and $(\widetilde{C}_2)$.*

(a) *If there exists a solution $\alpha > t_0$ of the equation $f'(t) = 0$, then α is the unique minimum of $f(t)$ in $[t_0, +\infty)$ and $f(t)$ is non-increasing in $[t_0, \alpha)$.*
(b) *If $f(\alpha) \leq 0$, then the equation $f(t) = 0$ has at least one solution in $[t_0, +\infty)$. Moreover, if t^* is the smallest solution of $f(t) = 0$ in $[t_0, +\infty)$, we have $t_0 < t^* \leq \alpha$.*

Taking into account the hypotheses of Theorem 14, function (6) satisfies the conditions of Theorem 9 and the semilocal convergence of Newton's method is then guaranteed in the Banach space X. In particular, we have the following theorem, whose proof follows immediately from Theorem 9.

Theorem 15. *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a nonlinear k $(k \geq 3)$ times continuously differentiable operator on a non-empty open convex domain Ω and $f(t)$ be the function defined in (6). Suppose that $(\widetilde{C}_1)$ and $(\widetilde{C}_2)$ hold, there exists a solution $\alpha > t_0$ of $f'(t) = 0$, such that $f(\alpha) \leq 0$, and $\overline{B(x_0, t^* - t_0)} \subseteq \Omega$, where t^* is the smallest root of $f(t) = 0$ in $[t_0, +\infty)$. Then, Newton's sequence $\{x_n\}$, given by (1), converges to a solution x^* of $F(x) = 0$ starting at x_0. Moreover, $x_n, x^* \in \overline{B(x_0, t^* - t_0)}$ and*

$$\|x^* - x_n\| \leq t^* - t_n, \quad \textit{for all } n = 0, 1, 2, \ldots,$$

where $t_n = N(t_{n-1}) = t_{n-1} - \frac{f(t_{n-1})}{f'(t_{n-1})}$, with $n \in \mathbb{N}$.

Remark 16. Note that the function given in (6) is such that $f(t + t_0) = \hat{f}(t)$, where

$$\hat{f}(t) = \int_0^t \int_0^{\theta_{k-1}} \cdots \int_0^{\theta_1} \omega(z; 0)\, dz\, d\theta_1 \ldots d\theta_{k-1} + \frac{M_{k-1}}{(k-1)!} t^{k-1} + \cdots + \frac{M_2}{2!} t^2 - \frac{t}{\beta} + \frac{\eta}{\beta}.$$

Therefore, the scalar sequences given by Newton's method with f and $\hat{f}$ can be obtained, one from the other, by translation. In consequence, the last results are independent of the value t_0. For this reason, we always choose $t_0 = 0$, which simplifies considerably the expressions used. Observe that Kantorovich's polynomial (4) has also this property, so that Kantorovich always considers $s_0 = 0$; see [12].

In [5], we can see an interesting work, where a similar study is presented for polynomial equations in Banach spaces. The author uses a different technique from Kantorovich's, which is based on some ideas given in [9].

5. Application: a conservative problem

We illustrate the study developed above with an application where a nonlinear conservative problem is involved. Our study improves that given by Kantorovich under conditions (C_1)–(C_3).

It is well known that energy is dissipated in the action of any real dynamical system, usually through some form of friction. However, in certain situations this dissipation is so slow that it can be neglected over relatively short periods of time. In such cases we assume the law of conservation of energy, namely, that the sum of the kinetic energy and the potential energy is constant. A system of this kind is said to be conservative.

If φ and ψ are arbitrary functions with the property that $\varphi(0) = 0$ and $\psi(0) = 0$, the general equation

$$\mu \frac{d^2x(t)}{dt^2} + \psi\left(\frac{dx(t)}{dt}\right) + \varphi(x) = 0, \tag{7}$$

can be interpreted as the equation of motion of a mass μ under the action of a restoring force $-\varphi(x)$ and a damping force $-\psi(dx/dt)$. In general these forces are nonlinear, and Eq. (7) can be regarded as the basic equation of nonlinear mechanics. In this paper we shall consider the special case of a nonlinear conservative system described by the equation

$$\mu \frac{d^2x(t)}{dt^2} + \varphi(x(t)) = 0,$$

in which the damping force is zero and there is consequently no dissipation of energy. Extensive discussions of (7), with applications to a variety of physical problems, can be found in classical Refs. [3,15].

In this paper, we study the existence of a unique solution for a special case of a nonlinear conservative system described by the equation

$$\frac{d^2x(t)}{dt^2} + \phi(x(t)) = 0 \tag{8}$$

with the boundary conditions

$$x(0) = x(1) = 0. \tag{9}$$

In order to study the application of (1) for the numerical solution of differential equation problems, we illustrate the theory for the case of particular second-order ordinary differential equation (8) subject to the boundary conditions (9).

It is required to find a solution of problem (8)–(9) in the interval $t \in [0, 1]$. Under suitable restrictions on the function ϕ, we will see that a unique solution of (8)–(9) exists. Moreover the method of discretization is used to project the boundary value problem of second order into a finite-dimensional space. The new family of secant-like methods are applied to this problem to approximate the solution of the corresponding system of equations.

For this, we consider the operator

$$[F(x)](t) = \frac{d^2x(t)}{dt^2} + \phi(x(t)), \tag{10}$$

which is an operator from $C^{(2)}[0, 1]$ into $C[0, 1]$. Then, solving problem (8) is equivalent to solving the equation $F(x) = 0$, where F is defined in (10).

Initially, we transform problem (8)–(9) into a finite dimensional problem. For this, we approximate the second derivative by a standard numerical formula. Moreover, from now on, we use the max-norm.

For the direct numerical solution of problem (8)–(9), we introduce the points $t_j = jh$, $j = 0, 1, \dots, m + 1$, where $h = \frac{1}{m+1}$ and m is an appropriate integer. A scheme is then designed for the determination of numbers x_j, it is hoped, approximate the values $x(t_j)$ of the true solution at the points t_j. A standard approximation for the second derivative at these points is

$$x_j'' \approx \frac{x_{j-1} - 2x_j + x_{j+1}}{h^2}, \quad j = 1, 2, \dots, m.$$

A natural way to obtain such a scheme is to demand that the x_j satisfy at each interior mesh point t_j the difference equation

$$x_{j-1} - 2x_j + x_{j+1} + h^2\phi(x_j) = 0. \tag{11}$$

Since x_0 and x_{m+1} are determined by the boundary conditions, the unknowns are $x_1, x_2, \dots, x_m$.

A further discussion is simplified by the use of matrix and vector notation. Introducing the vectors

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \qquad u_{\bar{x}} = \begin{pmatrix} \phi(x_1) \\ \phi(x_2) \\ \vdots \\ \phi(x_m) \end{pmatrix}$$

and the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix},$$

the system of equations, arising from demanding that (11) holds for $j = 1, 2, \dots, m$, can be written compactly in the form

$$F(\overline{x}) \equiv A\overline{x} + h^2 u_{\overline{x}} = 0, \tag{12}$$

which is a function from $\mathbb{R}^m$ into $\mathbb{R}^m$.

If the function $\phi(x)$ is not linear in x, we cannot hope to solve system (12) by algebraic methods. Some iterative procedure must be resorted to. Then, we analyze method (1) for this purpose.

From now on, we consider a particular case of (8). The steady temperature distribution is known in a homogeneous rod of length 1 in which, as a consequence of a chemical reaction or some such heat-producing process, heat is generated at a rate $\phi(x(t))$ per unit time per unit length, $\phi(x(t))$ being a given function of the excess temperature x of the rod over the temperature of the surroundings. If the ends of the rod, $t = 0$ and $t = 1$, are kept at given temperatures, we are to solve the boundary value problem given by (8) and (9), measured along the axis of the rod. For an example, we choose the following polynomial law

$$\phi(x(t)) = 3 + x(t) + 2x(t)^2 + x(t)^3 \tag{13}$$

for the heat generation.

According to the above-mentioned, $u_{\overline{x}} = (3 + x_1 + 2x_1^2 + x_1^3, \dots, 3 + x_m + 2x_m^2 + x_m^3)^T$, where $\overline{x} = (x_1, \dots, x_m)^T$. Consequently, the first derivative of the function F defined in (12) is given by

$$F'(\overline{x}) = A + h^2 D(v_{\overline{x}}),$$

where $v_{\overline{x}} = (1+4x_1+3x_1^2, \dots, 1+4x_m+3x_m^2)^T$ and $D(v_{\overline{x}}) = \operatorname{diag}\{1+4x_1+3x_1^2, \dots, 1+4x_m+3x_m^2\}$. Moreover,

$$F''(\overline{x})\overline{y}\,\overline{z} = (y_1, \dots, y_m)F''(\overline{x})(z_1, \dots, z_m),$$

where $\overline{y} = (y_1, \dots, y_m)^T$ and $\overline{z} = (z_1, \dots, z_m)^T$, so that

$$F''(\overline{x})\overline{y}\,\overline{z} = h^2((4 + 6x_1)y_1z_1, \dots, (4 + 6x_m)y_mz_m)^T.$$

Now, we study the application of Theorem 4, the Newton–Kantorovich theorem, to this problem. As

$$\|F''(\overline{x})\| = \sup_{\substack{\|\overline{y}\|=1 \\ \|\overline{z}\|=1}} \|F''(\overline{x})\overline{y}\,\overline{z}\|,$$

then

$$\|F''(\overline{x})\overline{y}\,\overline{z}\| \le h^2 \left\| \begin{pmatrix} (4 + 6x_1)y_1z_1 \\ \vdots \\ (4 + 6x_m)y_mz_m \end{pmatrix} \right\| \le h^2(4 + 6\|\overline{x}\|)\|\overline{y}\|\,\|\overline{z}\|.$$

Observe that in this case $\|F''(\overline{x})\|$ is not bounded in general, since the function $q(t) = 4 + 6t$ is increasing. Therefore, condition (C_2) of Kantorovich is not satisfied, so that we cannot apply Theorem 4.

To solve the last difficulty and apply Theorem 4, a common alternative is to locate a solution of Eq. (12) in some domain and look for a bound for $\|F''(\overline{x})\|$ there (see [8]). For this, taking into account that the solution of (8)–(9) with $\phi(x(t))$ defined in (13) is of the form [14]

$$x(t) = \int_0^1 G(t, \xi)\phi(x(\xi))\, d\xi,$$

Table 1
Numerical solution $\overline{x}^*$ of (12) with $\phi(x)$ defined in (13).

n	x^*_n	n	x^*_n
1	0.18110836 ...	5	0.46853793 ...
2	0.32206056 ...	6	0.41902620 ...
3	0.41902620 ...	7	0.32206056 ...
4	0.46853793 ...	8	0.18110836 ...

where $G(t, \xi)$ is the Green function in $[0, 1]$, we can locate the solution $x^*(t)$ in some domain. So, we have

$$\|x^*(t)\| - \frac{1}{8}\left(3 + \|x^*(t)\| + 2\|x^*(t)\|^2 + \|x^*(t)\|^3\right) \le 0,$$

where $\frac{1}{8} = \max_{[0,1]} \int_0^1 |G(t, \xi)|\, d\xi$, so that $\|x^*(t)\| \in [0, \sigma_1] \cup [\sigma_2, +\infty]$, where $\sigma_1 = 0.5301\ldots$ and $\sigma_2 = 1.4291\ldots$ are the two positive real roots of the scalar equation $t - \frac{1}{8}(3 + t + 2t^2 + t^3) = 0$.

By Theorem 4, we could only guarantee the semilocal convergence of Newton's method to a solution $x^*(t)$ such that $\|x^*(t)\| \in [0, \sigma_1]$. For this, we can consider the domain

$$\Omega = \left\{x(t) \in \mathcal{C}^{(2)}([0, 1]);\ \|x(t)\| < \frac{3}{4},\ t \in [0, 1]\right\}$$

since $\sigma_1 < \frac{3}{4} < \sigma_2$.

In view of what the domain Ω is for Eq. (8), we then consider (12) with $F : \Lambda \subset \mathbb{R}^m \to \mathbb{R}^m$ and $\Lambda = \left\{\overline{x} \in \mathbb{R}^m;\ \|\overline{x}\| < \frac{3}{4}\right\}$.

If we choose $m = 8$ and $\overline{x}_0 = (0, \ldots, 0)^T$ as a starting point, we see that conditions of Theorem 4 are satisfied, since

$$\|F''(\overline{x})\| \le \frac{17}{162} = \ell, \quad \beta = 11.1694\ldots, \quad \eta = 0.4136\ldots \quad \text{and} \quad \ell\beta\eta = 0.4848\ldots < \frac{1}{2}.$$

Moreover, as $p(s) = \frac{17}{324}s^2 - (0.0895\ldots)s + (0.0370\ldots)$, then $s^* = 0.7047\ldots$ and $B(\overline{x}_0, s^*) \subseteq \Lambda = B\left(\overline{x}_0, \frac{3}{4}\right)$. Therefore, Newton's method converges to the solution $\overline{x}^* = (x_1^*, \ldots, x_8^*)^T$ shown in Table 1 after eight iterations with tolerance 10^{-30}.

On the other hand, if we consider Theorem 15 with $t_0 = 0$ and take into account that $\|F''(\overline{x}_0)\| \le \frac{4}{81} = M_2$ and $\omega(t) = \frac{2}{27}$, we see that

$$f(t) = \frac{t^3}{81} + \frac{2}{81}t^2 - (0.0895\ldots)t + (0.0370\ldots), \tag{14}$$

so that $f(\alpha) = -0.0154\ldots < 0$, since $\alpha = 1.0250\ldots$. Consequently, the convergence of Newton's method to the solution $\overline{x}^*$ is also guaranteed from Theorem 15.

Next, for Theorem 4, we obtain $s^* = 0.7047\ldots$ and $s^{**} = 1.0015\ldots$, so that the domains of existence and uniqueness of solution are respectively

$$\{\overline{u} \in \mathbb{R}^8;\ \|\overline{u}\| \le 0.7047\ldots\} \quad \text{and} \quad \{\overline{u} \in \mathbb{R}^8;\ \|\overline{u}\| < 1.0015\ldots\}.$$

For Theorems 11 and 15, we obtain $t^* = 0.4997\ldots$ and $t^{**} = 1.5005\ldots$. In addition, the domains of existence and uniqueness of solution are respectively

$$\{\overline{u} \in \mathbb{R}^8;\ \|\overline{u}\| \le 0.4997\ldots\} \quad \text{and} \quad \{\overline{u} \in \mathbb{R}^8;\ \|\overline{u}\| < 1.5005\ldots\}.$$

We then see that the domains obtained from Theorem 4 are improved by Theorems 11 and 15, since the domain of existence of solution is smaller and the domain of uniqueness of solution is bigger.

If we interpolate the values given in Table 1 and take into account (9), we obtain the solution drawn in Fig. 1 and denoted by $\widetilde{x}$. Observe that $\|\overline{x}^*\| = 0.4685\ldots \le \frac{3}{4}$.

After that, we see that the corresponding majorizing sequence given by (5) provides better a priori error estimates than those obtained from the majorizing sequence $\{s_n\}$ obtained from corresponding

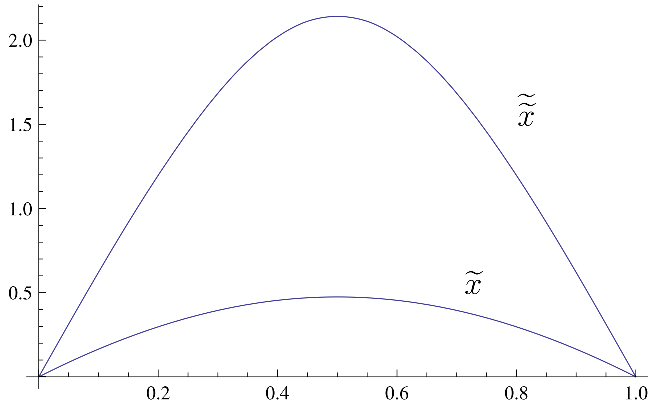

Fig. 1. Approximated solutions of (8)–(9) with $\phi(x(t))$ defined in (13).

Table 2
Absolute error and a priori error estimates.

n	$\|\bar{x}^* - \bar{x}_n\|$	$\|t^* - t_n\|$	$\|s^* - s_n\|$
0	0.46853793 ...	0.49978515 ...	0.70479160 ...
1	0.05485523 ...	0.08610244 ...	0.29110889 ...
2	0.00119680 ...	0.00485193 ...	0.09641174 ...
3	$6.07157854\ldots \times 10^{-7}$	0.00001810 ...	0.01898384 ...

polynomial (4). The a priori error estimates and the absolute error are shown in Table 2. Observe the remarkable improvement obtained from the new majorizing sequence constructed in this paper.

On the other hand, we have seen previously that problem (8)–(9) with $\phi(x(t))$ defined in (13) may have a solution $x^{**}(t)$ such that $\|x^{**}(t)\| \geq \sigma_2 = 1.4291\ldots$, but we cannot guarantee the convergence of Newton's method from Theorem 4, since we cannot fix a domain that contains $x^{**}(t)$ and where $\|F''(x)\|$ is bounded. However, from Theorem 15, we can do it.

We choose for example the starting vector $\bar{x}_0 = (2, \ldots, 2)^T$ and observe that $\|\bar{x}_0\| = 2 > \sigma_2 = 1.4291\ldots$. At first, we cannot apply Theorem 15 either, since $f(\alpha) = 0.1157\ldots > 0$ with $\alpha = 0.4952\ldots$ and

$$f(t) = \frac{t^3}{81} + \frac{8}{81}t^2 - (0.1069\ldots)t + (0.1430\ldots),$$

$\beta = 9.3529\ldots$, $\eta = 1.3375\ldots$ and $M_2 = \frac{16}{81}$. However, it seems clear that improving the initial approximation, the conditions of Theorem 15 hold. Indeed, applying Newton's method from $\bar{x}_0 = (2, \ldots, 2)^T$, after two iterations, we obtain the vector $\bar{x}_2$ given by

$$\bar{x}_2 = \begin{pmatrix} 0.70391713\ldots \\ 1.34565183\ldots \\ 1.85928293\ldots \\ 2.15200198\ldots \\ 2.15200198\ldots \\ 1.85928293\ldots \\ 1.34565183\ldots \\ 0.70391713\ldots \end{pmatrix},$$

that satisfies, as new starting point, the conditions of Theorem 15, since if we take $\tilde{\tilde{x}}_0 = \bar{x}_2$, we obtain $\beta = 11.4307\ldots$, $\eta = 0.0453\ldots$, $M_2 = 0.2087\ldots$,

$$f(t) = \frac{t^3}{81} + (0.1043\ldots)t^2 - (0.0874\ldots)t + 0.0039\ldots,$$

Table 3
Numerical solution $\overline{x}^{**}$ of (12) with $\phi(x)$ defined in (13).

n	x_n^{**}	n	x_n^{**}
1	0.68796551...	5	2.10501296...
2	1.31469436...	6	1.81742466...
3	1.81742466...	7	1.31469436...
4	2.10501296...	8	0.68796551...

Table 4
Absolute error and a priori error estimates.

n	$\Vert\overline{x}^{**} - \overline{x}_n\Vert$	$\vert t^* - t_n\vert$
0	0.04698902...	0.04811926...
1	0.00164855...	0.00277878...
2	$2.0311\ldots \times 10^{-6}$	0.00001051...
3	$3.1208\ldots \times 10^{-12}$	$1.5186\ldots \times 10^{-10}$

$\alpha = 0.3917\ldots$ and $f(\alpha) = -0.0135\ldots \leq 0$. Besides, $t^* = 0.0481\ldots$, $t^{**} = 0.7235\ldots$ and the domains of existence and uniqueness of solution are respectively

$$\{\overline{u} \in \Lambda;\ \|\overline{u} - \widetilde{\widetilde{x}}_0\| \leq 0.0481\ldots\} \quad \text{and} \quad \{\overline{u} \in \Lambda;\ \|\overline{u} - \widetilde{\widetilde{x}}_0\| < 0.7235\ldots\}.$$

Observe that the new starting point $\widetilde{\widetilde{x}}_0$ is such that $\|\widetilde{\widetilde{x}}_0\| = 2.1520\ldots > \sigma_2 = 1.4291\ldots$. After seven more iterations of Newton's method, we obtain the approximated solution $\overline{x}^{**} = (x_1^{**}, \ldots, x_8^{**})^T$ given in Table 3, which is a solution that is beyond the scope of the Newton–Kantorovich theorem (Theorem 4). Observe that $\|\overline{x}^{**}\| = 2.1050\ldots > \sigma_2 = 1.4291\ldots$.

For the solution $\overline{x}^{**}$, we can find in Table 4 the a priori error estimates and the absolute error.

By interpolating the values of Table 3 and taking into account (9), we obtain the solution drawn in Fig. 1 and denoted by $\widetilde{\widetilde{x}}$.

Remark 17. We know that nonlinear differential equations of the second order can be reduced to simpler ones by different transformations. The prototype of some of them is one of the earliest examples known, namely, the equation which describes the oscillation of the simple pendulum:

$$\frac{d^2x(t)}{dt^2} + c^2 \sin x(t) = 0,$$

which is usually approximated by

$$\frac{d^2x(t)}{dt^2} + Cx(t) = 0.$$

A generalization of this idea could be considered here. For example, if the power series for $\exp(x)$ is truncated, Eq. (8) with $\phi(x(t)) = \exp(x(t))$ can be written in the form

$$\frac{d^2x(t)}{dt^2} + \sum_{i=0}^{j} \frac{x(t)^i}{i!} = 0.$$

Taking now into account the operator

$$[F(x)](t) = \frac{d^2x(t)}{dt^2} + \sum_{i=0}^{j} \frac{x(t)^i}{i!},$$

it is clear that the operator $F^{(j)}(x)$ is constant, so that we can then consider this kind of interesting approximations, since the polynomial operators so obtained always satisfy a condition of type $(\widetilde{C}_2)$, where the function ω is constant.

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